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**Effects of amplitude nullification in the standard
model**

B.H. Smith

School of Physics and Astronomy
University of Minnesota
Minneapolis, MN 55455

Abstract

Delicate cancellations among diagrams can result in the vanishing of threshold amplitudes in the standard model. This phenomenon is investigated for multi-Higgs production by scalar, vector, and fermionic fields. A surprising gap is found where although the tree-level amplitude for 2 incoming gauge bosons to produce N_v particle may vanish, the amplitude for the production of $N_v + 1$ particles is nonzero.

One of the most interesting pieces of phenomenology to come from the study of multi-particle processes is amplitude nullification^[1, 2, 3]. The interest in this phenomenon was spurred by an observation about the tree-level amplitude for two on-shell scalar particles scattering to produce n particles. At the kinematical threshold, this $2 \rightarrow n$ amplitude vanishes for all $n > 4$ in $\lambda\phi^4$ theory with an unbroken reflection symmetry^[1]. In $\lambda\phi^4$ theory with broken symmetry, it was noted that the tree-level threshold amplitude of $2 \rightarrow n$ production vanishes for all $n > 2$ ^[3].

The nullification phenomenon forbids the threshold amplitude for one off-shell scalar decaying into n scalars, $1 \rightarrow n$, from developing an imaginary part at the one-loop level. At the one-loop level, any relevant unitary cut must cross exactly two lines. Any such cut will relate the imaginary part of the amplitude to the product of the vanishing $2 \rightarrow n$ amplitude.

At the threshold, amplitudes of $1 \rightarrow n$ processes still exhibit factorial growth, and, therefore, appear to violate unitarity. It is not clear what happens above the threshold, and it may be possible that diagram cancellation plays a critical role in the restoration of unitarity. It can be speculated that this nullification is the manifestation of some as yet undiscovered symmetry. There exists a scenario where nullification could provide a clue as to the restoration unitarity at high multiplicity in the standard model^[4]. In order for this to occur, the coupling of the Higgs field to the other fields in the standard model would have to take on very special values. Requiring these special couplings to occur would place restrictions on the ratios of masses in the standard model, and be an attractive way of explaining the particle mass spectrum.

For simplicity, we will consider a $U(1)$ gauge theory coupled to a fermionic field and a complex scalar field. All of the features in the following discussion can be easily generalized to the standard model. The masses of the gauge and fermionic fields are generated in the standard way through the Higgs mechanism. The mass and quadratic coupling of the physical scalar particle shall be denoted as m and $\lambda/4$, respectively. After symmetry breaking, the scalar field develops a vacuum expectation value $v = \sqrt{m^2/(2\lambda)}$. For simplicity, we will work in units where $m = 1$. The amplitude calculations shall rely on the reduction formula technique^[5] for calculating multi-particle amplitudes. The key observation for using this technique, is that at the threshold, all fields have zero momenta and are functions of time only. The LSZ reduction simplifies to a one dimensional problem. Using this method, the amplitude for the threshold decay of one scalar to n scalars is,

$$\langle n|\sigma|0\rangle = \left(\frac{\partial}{\partial z}\right)^n \langle 0 + |\sigma(x)|0-\rangle, \quad (1)$$

where $z(t)$ is the response of the free Higgs field in the presence of a source, ρ , and is proportional to e^{imt} , and $\langle 0 + |\sigma|0-\rangle$ is the response of the interacting Higgs field in the presence of a source. The tree-level amplitudes are generated by approximating the response by the solution to the classical field equations, $\sigma_{cl}(t)$. This well known generating function for $1 \rightarrow n$ processes is^[5],

$$\sigma_{cl}(z(t)) = \frac{z(t)}{1 - z(t)\sqrt{\frac{\lambda}{2m^2}}}. \quad (2)$$

The first loop correction to the amplitude is generated by expanding the mean value of the field around the classical solution, $\langle 0 + |\sigma(x)|0-\rangle = \sigma_{cl}(x) + \sigma_1(x)$, where $\sigma_1(x)$ is the mean value of the quantum part of the field. The equations of motion can be expanded around $\sigma_{cl}(x)$, and only the leading contribution in λ needs to be retained at the one-loop level,

$$\begin{aligned} \frac{\partial^2 \sigma_1(t)}{\partial t^2} + m^2 \sigma_1(t) + 6m\sqrt{\frac{\lambda}{2}}\sigma_{cl}\sigma_1(t) + 3\lambda\sigma_{cl}^2\sigma_1(t) + \\ V'_{scalar}(t) + V'_{fermion}(t) + V'_{vector}(t) = 0, \end{aligned} \quad (3)$$

where the three functions, $V'(t)$, represent the contributions to the amplitude due to scalar, fermion, and vector field loops. Each of the contributing terms, $V(t)$, in equation (3) contains a term representing the equal time two-point field correlation function evaluated in the σ_{cl} background field. This can be visualized as the evaluation of a tadpole Feynman diagram, with the equal time Green function coming from the propagator in the loop (for a discussion on using diagrammatic techniques in the calculation of multiparticle processes, see [6]).

This two point Green function can be found by taking the inverse of the second variation of the Lagrangian evaluated with $\sigma = \sigma_{cl}$. The Green function is a transformation of the amplitude of 2 incoming particles to produce n scalars at the threshold. For physical processes, the interesting object is the double pole of the amplitude as the momenta of the incoming particles are taken to be on mass shell. These processes can be characterized by a single parameter, $\omega = \sqrt{\mathbf{k}^2 + 1} = n/2$, where \mathbf{k} is the spatial momentum of the incoming particles, and n is the number of particles produced

in the final state. If the Green function does not have a pole for $\omega = n/2$, then the tree-level amplitude for $2 \rightarrow n$ particles is zero.

In the following sections, we will discuss how particles from each sector of the standard model contribute to the one-loop amplitude of 1 Higgs $\rightarrow n$ Higgs and where nullification in the production of Higgs particles occurs in this sector.

The Higgs Sector

Both the one-loop correction to $1 \rightarrow n$ processes and $2 \rightarrow n$ nullification for incoming scalars have been discussed previously in [3]. The first variation of the scalar potential contributes terms to the equation of motion that are both quadratic and cubic in the scalar field, σ . Expanding around the mean field,

$$V'_{scalar}(t) = 3\sqrt{\frac{\lambda}{2}}\langle\sigma_q(x)\sigma_q(x)\rangle + 3\lambda\sigma_{cl}\langle\sigma_q(x)\sigma_q(x)\rangle, \quad (4)$$

where $\langle\sigma_q(x)\sigma_q(x)\rangle$ is the spatially independent two-point Green function in the classical background field, $G(x, x') \equiv \langle T\sigma_q(x)\sigma_q(x')\rangle$, taken in the limit $x \rightarrow x'$. As discussed in a previous work^[3], the Green function is the inverse of the second variation of the Lagrangian evaluated in the classical background field.

$$\left[\partial^2 + 1 + 6\sqrt{\frac{\lambda}{2}}\sigma_{cl}(t) + 3\lambda\sigma_{cl}(t)^2\right]G(x, x') = -i\delta(\tau, \tau'). \quad (5)$$

The component of $G(x, x')$ with spatial momentum \mathbf{k} , denoted as $G_\omega(\tau, \tau')$ has been constructed from the homogeneous equation with the operator in equation (5)^[3]. For future reference, we express $G_\omega(\tau, \tau')$ in terms of a general Green function,

$$G_\omega(a, \tau, \tau') = 1/(2\omega)(f_a^+(\tau)f_a^-(\tau')\Theta(\tau - \tau') + f_a^-(\tau)f_a^+(\tau)\Theta(\tau' - \tau)), \quad (6)$$

where τ is defined by,

$$e^\tau = -\sqrt{\frac{\lambda}{2}}z(t), \quad (7)$$

and the two solutions to the homogenous differential equation with the operator in equation (5) are

$$f_a^\pm = e^{\mp\omega\tau}F(-a, a+1; 1 \pm 2\omega; \frac{1}{1+e^\tau}), \quad (8)$$

and $F(a, b; c; z)$ is the hypergeometric function. The hypergeometric series in equation (8) will terminate when a is an integer. For incoming scalar particles, $G_\omega(\tau, \tau') = G_\omega(2, \tau, \tau')$. If the series did not terminate, the Green function would have poles for all half integer values of ω . Since the series terminates, there are poles only for $\omega = \pm 1/2, \pm 1$. Thus, the tree-level amplitude for two on-shell scalars to form n scalars vanishes at the threshold for $n > 2$.

Summing over all momenta modes, the equal time Green function is, aside from a renormalization of m and λ ,

$$\langle \sigma_q(x) \sigma_q(x) \rangle = \frac{\sqrt{3} \lambda z(t)^2}{4\pi(1 - z(t)\sqrt{\lambda/2})^4}. \quad (9)$$

The Higgs sector cannot contribute an imaginary part to the one-loop $1 \rightarrow n$ amplitude. There is no two body phase space when $\omega = 1/2, 1$ so it is not possible to make a nonzero unitary cut to contribute to the imaginary part.

Fermionic Contribution

The fermionic contribution to the equation of motion is found by varying the fermionic portion of the Lagrangian with respect to σ . As in the case of the scalar field, the mean value of σ is expanded around the classical solution, $\sigma_{cl}(t)$. The mean value of all other fields is set to zero. The fermionic term in equation (3) is,

$$V'_{fermion}(t) = \frac{m_f}{v} \langle \bar{\psi}(x) \psi(x) \rangle, \quad (10)$$

where $\langle \bar{\psi}(x) \psi(x) \rangle$ is the trace two-point fermionic Green function, $\langle T\bar{\psi}(x) \psi(x') \rangle$, and m_f is the mass of the fermion field as generated by the Higgs mechanism. The Green function has been discussed in [7] and can be found by calculating the inverse of the variation of the Lagrangian with respect to $\psi(x)$ and $\bar{\psi}(x)$ evaluated in the classical background field,

$$\left[i \partial - m_f - \frac{m_f}{v} \sigma_{cl}(t) \right] \langle T\bar{\psi}(x) \psi(x') \rangle = -i \delta^4(x - x'). \quad (11)$$

The spinor structure of equation (11) is simplified by the introduction of the Green function $H(x, x')$ defined by,

$$\langle T\bar{\psi}(x) \psi(x') \rangle = \left[-i \partial - m_f - \frac{m_f}{v} \sigma_{cl}(t) \right] H(x, x'), \quad (12)$$

and analytic continuation into imaginary time as in the previous section. Since spatial momentum is conserved, the Green function, $H(x, x')$, can be constructed from the solution to the homogeneous differential equation,

$$\left[\frac{\partial^2}{\partial \tau^2} - \omega^2 + \left(m_f^2 - \frac{m_f}{2} \gamma_0 \right) \frac{1}{\cosh^2 \tau/2} \right] f(\tau) = 0 \quad (13)$$

The matrix γ_0 in equation (13) has eigenvalues of $+1$ and -1 . As was noted by Voloshin in [7], in order for nullification to take place, the coefficient of the $\text{sech}^2(\tau/2)$ term must be $N(N+1)/4$ with integer N for both eigenvalues. This imposes the conditions on m_f ,

$$4m_f^2 + 2m_f = N_f(N_f + 1), \quad (14)$$

and,

$$4m_f^2 - 2m_f = N'_f(N'_f + 1), \quad (15)$$

for some integer values of N_f and N'_f . Both equation (14) and equation (15) are satisfied when $2m_f = N_f$ and $N'_f = N_f - 1$.

In a basis where the matrix H is diagonal, two of the diagonal elements of H will be $G_\omega(N_f, \tau, \tau')$ and two will be $G_\omega(N_f - 1, \tau, \tau')$. The fermionic Green function for a general fermionic coupling is,

$$\begin{aligned} \langle \bar{\psi}(x)\psi(x) \rangle &= 2 \lim_{\tau' \rightarrow \tau} \int \frac{d^3k}{(2\pi)^3} \left[-\frac{\partial}{d\tau} G_\omega(N_f, \tau, \tau') + \frac{\partial}{d\tau} G_\omega(N_f - 1, \tau, \tau') \right] - \\ &\quad 2 \lim_{\tau' \rightarrow \tau} \int \frac{d^3k}{(2\pi)^3} \left[\left(m_f + \frac{m_f}{v} \sigma_{cl}(t) \right) (G_\omega(N_f, \tau, \tau) + G_\omega(N_f - 1, \tau, \tau)) \right]. \end{aligned} \quad (16)$$

Equation (16) contains two types of terms. One type of terms contributes only to the renormalization of m and λ . The other terms actually contribute to the physical correction to the $1 \rightarrow n$ amplitude. In the theory with $N_f = 1$ ($m_f = m_h/2$), only the contribution to the renormalization of the coupling constant and mass is present.

The theory with the simplest nontrivial $\langle \bar{\psi}(x)\psi(x) \rangle$ and diagram nullification is characterized by $N_f = 2$, i.e. $m_f = m_h$. In this theory, the contributions from $G_\omega(N_f - 1, \tau, \tau')$ contribute only to the renormalization of m_h and λ . Apart from

renormalization terms, the fermionic correlator arising from $G_\omega(2, \tau, \tau')$ is,

$$\langle \bar{\psi}(x)\psi(x) \rangle = -\frac{\sqrt{3}\lambda z(t)^2(1+z(t)\sqrt{\lambda/2})}{3\pi^2(1-z(t)\sqrt{\lambda/2})^5}. \quad (17)$$

It is worth noting that the fermionic contribution to the equations of motion and, hence, to the one-loop correction is exactly $-\frac{2}{3}$ that of the scalar contribution.

Gauge Field Sector

The situation with vector fields is more complicated. The qualitative behavior of 2 vector $\rightarrow n$ scalar processes is very dependant on the polarization of the incoming gauge particles. Care must be taken in the choice of gauge. Nonphysical amplitudes like those of the $1 \rightarrow n$ processes discussed in previous sections are gauge dependent. While it is easiest to calculate in unitary gauge, the gauge loop corrections to processes like $1 \rightarrow n$ are divergent. These divergences will cancel when a gauge independent physical amplitude is calculated. We will work in unitary gauge for simplicity of calculation, but we will concentrate on discussing the properties of the physical amplitude of two incoming vector bosons scattering to form many scalars.

The two point Green function is calculated as in previous sections by taking the inverse of the second variation of the Lagrangian. In the unitary gauge, the two point vector Green function, $\langle V_\rho(x)V_\nu(x') \rangle$, satisfies the condition,

$$\left[\partial^2 + \frac{m_v^2}{v^2}(\sigma_{cl}(t) + v)^2 \right] \langle V_\mu(x)V_\nu(x') \rangle - \partial^\mu \partial^\nu \langle V_\rho(x)V_\nu(x') \rangle = -ig_{\mu\nu}\delta^4(x - x'). \quad (18)$$

There are no additional ghost contributions in unitary gauge. It is convenient to rotate to a frame of reference where the spatial momentum is along the "1" axis. The spatially fourier transformed Green function,

$$G_{\mu\nu}(t, t') = \int \langle V_\mu(x)V_\nu(x') \rangle e^{i\mathbf{k}\mathbf{x}} d^3\mathbf{x}, \quad (19)$$

has only two nonvanishing derivatives, ∂_0 and ∂_1 . For the transverse components, G_{22} and G_{33} , the last term on the left hand side of equation (18) does not contribute. The Green functions may be written as in previous sections in terms

of the solutions to the homogeneous differential equation constructed from the operator in equation (18),

$$G_{\mu\nu} = \frac{1}{W}(f_{\mu\nu}^+(\tau)f_{\mu\nu}^-(\tau')\Theta(\tau - \tau') + f_{\mu\nu}^-(\tau)f_{\mu\nu}^+(\tau')\Theta(\tau - \tau')), \quad (20)$$

Where W is the Wronskian of the two solutions. In analogy with previous sections, we define the variable, N_v , which satisfies,

$$4m_v^2 = N_v(N_v + 1). \quad (21)$$

For the transverse components,

$$f_{22}^\pm = f_{33}^\pm = e^{\omega\tau} F(-N_v, N_v + 1, 1 \mp 2\omega; \frac{1}{1 + e^\tau}). \quad (22)$$

The Wronskian for these solutions is 2ω . The transverse part of the Green function will have poles at all half integer values of ω unless N_v is an integer. As noted by Voloshin^[7], the physical tree-level amplitude for 2 incoming transversely polarized vector particles to produce n scalars at the threshold vanishes for all $n > N_v$. The transverse off-diagonal elements of $G_{\mu\nu}$ are zero.

For longitudinally polarized particles, things are a little bit more complicated. Differentiation of equation (18) yields the auxillary equation,

$$\partial_\mu \frac{M_v^2 \tanh^2 \tau/2}{v^2} G_{\mu\nu} = \partial_\nu \delta(\tau - \tau'). \quad (23)$$

Searching for solutions of the form,

$$G_{00}(\tau, \tau') = \frac{1}{\tanh \tau \tanh \tau'} H(\tau, \tau') - \frac{v^2}{M_V^2 \tanh^2 \tau/2} \delta(\tau - \tau'), \quad (24)$$

the Green function, $H(\tau, \tau')$, becomes the object of interest. Combining equations (23) and (18) with the definition of $H(\tau, \tau')$,

$$\left[\frac{d}{d\tau^2} - \omega^2 - m_v^2 \tanh^2(\tau/2) - \frac{1}{2 \sinh^2 \tau/2} \right] H(\tau, \tau') = \frac{(\omega^2 - 1)v^2}{M_v^2} \delta(\tau - \tau'). \quad (25)$$

As in equation (20), $H(\tau)$ can be constructed from the two solutions to the homogenous differential equation with the operator in equation (25). The operator is the same as the one that appears in the Pöschl-Teller equation. The solution as described in the Appendix is,

$$f_H^\pm = \left[-2 \frac{d}{d\tau} + a \tanh(\tau/2) + \coth(\tau/2) \right] e^{\mp \omega \tau} F(-(N_v - 1), N_v; 1 \pm 2\omega; \frac{1}{1 + e^\tau}). \quad (26)$$

Note that the normalization of the Green function, $H(\tau, \tau')$, on the right hand side of equation (25) is not unity. As with the case of transverse gauge particles, there are poles in f_H at every half integer value of ω when N_v is not an integer. When N_v is an integer, there are no poles for $\omega > (N_v - 1)/2$. The Wronskian of the functions f_H^\pm can be conveniently evaluated at the point $\tau = \infty$. The result is

$$-\omega [4\omega^2 - (N_v + 1)^2]. \quad (27)$$

There is an additional pole contributed by the Wronskian at the point $\omega = (N_v + 1)/2$. The unusual pattern of the nullification is the following: all process are allowed for $n < N_v$. The production of $n = N_v + 1$ (which is forbidden in the transverse mode) is allowed. There is a small gap in the allowed region, when N_v particles are produced, which is not allowed. Production of n particles for $n > N_v + 1$ is forbidden. The remaining components of $G_{\mu\nu}$ can be easily constructed from the above formulae, and have the same poles as G_{00} .

Because of the pole at $n = N_v + 1$, gauge field loops can contribute an imaginary part to $1 \rightarrow n$ amplitude. The amplitude for the $1 \rightarrow n$ process is not a gauge invariant object, and is highly divergent in the unitary gauge.

Concluding Remarks

The phenomenology associated with nullification can be quite complex. Its presence can prevent the one-loop $1 \rightarrow n$ amplitude from developing an imaginary part. It can also cause the cancellation of $2 \rightarrow n$ diagrams in all sectors of the standard model for $n > N$ for some value of N .

The behavior in the longitudinal vectors is quite unexpected. While for incoming transverse vectors there is cancellation of $2 \rightarrow n$ diagrams for all $n > N_v$, the process with $n = N_v + 1$ is allowed for longitudinal polarized particles. In addition, the process with $n = N_v$ is forbidden for longitudinal polarized particles. The pole at $n = N_v + 1$ is what allows the $1 \rightarrow n$ amplitude to develop an imaginary part. It is of interest to note that the number of poles is the same regardless of the polarization of the incoming particles.

Although there exists much knowledge about the phenomenology of nullification, little is known about its root cause. It is possible that it is associated with some symmetry in the standard model that may play a part in the restoration of unitarity at large multiplicities. However, other scenarios have been suggested^[8].

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Appendix: Solution of the second Pöschl-Teller Equation through the methods of supersymmetric quantum mechanics

In the section on vector fields, the Green function was related to the solution to the Pöschl-Teller equation, which has the form,

$$\left[\frac{d^2}{dx^2} + \frac{a(a+1)}{\cosh^2(x)} - \frac{b(b+1)}{\sinh^2(x)} \right] f = \lambda f. \quad (28)$$

For the Green functions of vector fields, b has the value of 1.

This problem may be elegantly solved by using supersymmetric arguments (many reviews have been written on the uses of supersymmetry in quantum mechanics, e.g. [9]). Consider supersymmetric quantum mechanics with two supercharges defined to be,

$$Q_{\pm} = [\pm \frac{d}{dx} + W(x)]\sigma^{\pm} \quad (29)$$

where σ denotes the appropriate 2x2 Pauli spin matrix and the choice of potential is made to be,

$$W(x) = a \tanh(x) + b \coth(x) \quad (30)$$

The supersymmetric Hamiltonian is the anticommutator of the two supercharges,

$$H = \frac{1}{2}\{Q_+, Q_-\} = -\frac{d^2}{dx^2} + (a+b)^2 - \frac{a}{\cosh^2(x)}(a - \sigma_3) + \frac{b}{\sinh^2(x)}(b - \sigma_3). \quad (31)$$

This operator σ_1 had two eigenvalues, +1 and -1. These degenerate eigenstates shall be denote as ψ_{\pm} , and obey the eigenvalue equations,

$$H^{\pm}\psi^{\pm} \equiv \left[-\frac{d^2}{dx^2} + (a+b)^2 - \frac{a(a \mp 1)}{\cosh^2(x)} + \frac{b(b \mp 1)}{\sinh^2(x)} \right] \psi^{\pm} = E\psi^{\pm}. \quad (32)$$

Consider the case $b = 1$. The second term in H^+ disappears to leave a familiar operator. The solutions are known to be,

$$\psi^+(x) = e^{\pm\omega x} F(-(a-1), a; 1 \mp \omega; \frac{1}{1+e^x}), \quad (33)$$

where $\omega = \sqrt{E - (a+1)^2}$, and $F(a, b; c; z)$ is the hypergeometric function. By virtue of the superymmetry, the eigenvalues of H^- are simply $Q_-\psi^+$. It is easy to see the

the solution to equation (28) for $b = 1$ is simply,

$$f = \left[-\frac{d}{dx} + a \tanh(x) + \coth(x) \right] e^{\pm\sqrt{\lambda}} F(-(a-1), a; 1 \mp \sqrt{\lambda}; \frac{1}{1+e^x}). \quad (34)$$

Solutions to equation (28) for larger integer values of b may be constructed by iterating this procedure.

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